MODULATORS USING LINEAR FM INPUT SIGNALS

Lei Shang and Zahir M. Hussain, SMIEEE

School of Electrical and Computer Engineering
RMIT University, Melbourne, Victoria 3000, Australia
Emails: s9612136@student.rmit.edu.au, zmhussain@ieee.org

ABSTRACT

In this paper we present a simple and intuitive way to analyze different aspects of digital Delta and Sigma-Delta modulators. The analysis is based on studying the spectrum of the signals at different stages of the modulator. Linear FM (LFM) signals are used as test signals for different configurations. The evaluation of systems is simplified to "spectrum matching" between the input signal sequence and the output sequence of the quantizer. "Noise shaping" is also analyzed for the verification of the linear model assumption.

1. INTRODUCTION

Sigma-Delta ($\Sigma\Delta$) modulation based analog-to-digital (A/D) conversion technology is a cost effective alternative for high-resolution applications. Oversampling eases analog filter design, and also generates a spectrum with quantization noise pushed towards higher frequencies which are inaudible in audio applications. Over the last few years $\Sigma\Delta$ analogue-to-digital converters (ADCs) and digital-to-analogue converters (DACs) have become widely available, particularly for low-frequency applications such as high-fidelity audio and speech processing, metering applications, and voiceband data telecommunications [1] [2] [3].

Digital $\Sigma\Delta$ systems are easy for implementation and analysis. Analysis of $\Sigma\Delta$ modulation in the z-domain conventionally involves the assumption od a linear model in which the quantizer is modelled as Additive White Gaussian Noise (AWGN) source. In some circumstances this white noise assumption is not valid [4]. The input signal applied normally is a sinusoid or a DC. We present here an analysis by applying a Linear FM (LFM) signal as the input signal. The LFM possesses all possible frequencies with the same magnitude in the bandwidth of interest. All signals used in our analysis will be discrete sequences.

The oversampling process in $\Sigma\Delta$ modulators improves the resolution of a Nyquist rate data converter. This improvement is achieved by sampling the input signal at a significantly faster rate than the Nyquist rate. The ratio between the sampling rate and two times the signal bandwidth is defined as the oversampling ratio (OSR) [4]

$$\mathrm{OSR} = \frac{f_s}{2f_b} \tag{1}$$

where f_s is the sampling rate and f_b is the signal bandwidth. Apparently, $\mathrm{OSR} = 1$ means sampling at the Nyquist rate. A system with $\mathrm{OSR} = 1$ is generally called a Nyquist-rate system, while a system with $\mathrm{OSR} \gg 1$ is called an oversampling system. In this analysis of $\Sigma\Delta$ systems, the bandwidth of the LFM applied signal is approximately 2 Hz and the sampling rate is 100 Hz, hence $\mathrm{OSR} = 25$.

A discrete-time LFM is given by [5]

$$x[n] = \cos(\omega_o n^2) \tag{2}$$

whose instantaneous frequency (IF) is $2\omega_o n$.

The z-transform of a sequence $x[n]$ is defined as [5]

$$X(z) = \mathcal{Z}(x[n]) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}. \tag{3}$$

The z-transform evaluated on the unit circle corresponds to the discrete Fourier transform (DFT). As for practical digital Δ or $\Sigma\Delta$ systems, all signals are finite-duration sequences, it is convenient to find their spectra by DFT. We plot the signals in the continuous-time domain for demonstration purposes.

2. SPECTRAL ANALYSIS OF THE DELTA MODULATOR

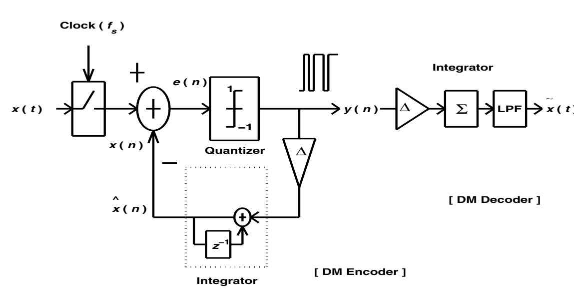

Fig. 1. First-order digital Δ modulator and demodulator.

Consider the 1-bit digital Δ modulator (encoder) and demodulator (decoder) shown in Fig. 1. A sinusoid of normalized frequency is applied to the system as a test signal. Fig. 2 shows the signals in the time domain and their corresponding spectra evaluated at the upper half of the unit circle by z-transform for the configuration shown in Fig. 1. Using this oversampled $\Sigma\Delta$ technique, the sinusoid is represented as a

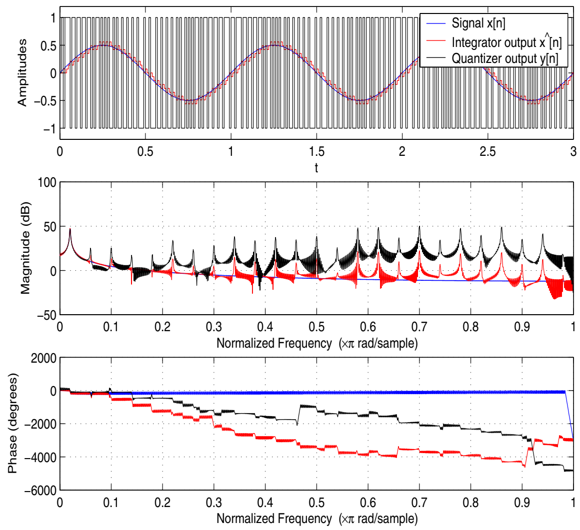

Fig. 2. Quantizing a sinusoid by the Δ modulator: $x[n]$ is the input signal, $\hat{x}[n]$ is its estimate, and $y[n]$ is the output single-bit stream. The corresponding spectra are also shown.

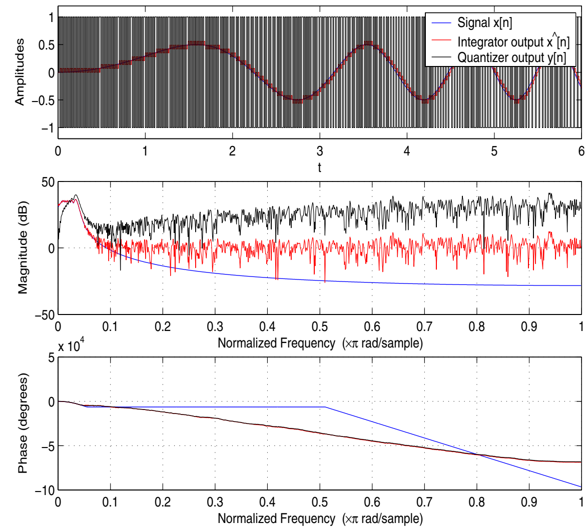

Fig. 3. Quantizing an LFM signal by the Δ modulator: $x[n]$ is the input signal, $\hat{x}[n]$ is its estimate, and $y[n]$ is the output single-bit stream. The corresponding spectra are also shown.

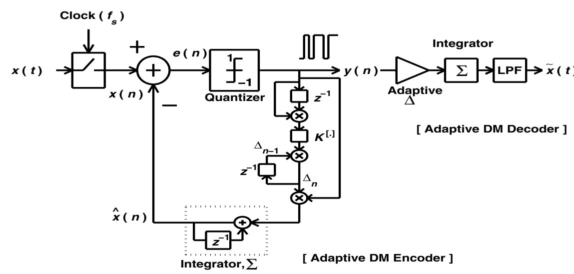

Fig. 4. First-order digital adaptive Δ modulator and demodulator.

single-bit binary data stream $\in \{-1, 1\}$. For the plots of magnitude and phase in Fig. 2, the value of 1 in the frequency axis corresponds to $f_s/2$. In this case, OSR $= 50$. We see in this trivial scenario that, as long as no slope-overload distortion occurs, the output of the quantizer $y[n]$, which is a single-bit binary data stream $\in \{-1, 1\}$, preserves the spectrum of the original signal in the bandwidth of interest. The spectrum obtained by DFT of this binary data stream $\in \{-1, 1\}$ is approximately equal to the spectrum of the input sequence $x[n]$ over baseband, and contains noise outside of baseband.

Now we use LFM as the input signal. The quantization process is shown in Fig.3. We see that the spectrum of $y[n]$ is distorted as compared to the spectrum of $x[n]$. The Δ modulator (encoder) is sensitive to the sampling frequency and the quantization step. An integrator is necessary in the demodulator (decoder) to reconstruct the signal to $\hat{x}[n]$, as shown in Fig. 2, i.e, a step amplifier and an integrator are needed in the demodulator before low-pass filtering (LPF). We also notice that the phase is also "matched" in the bandwidth of interest.

Our design of the encoder focuses on matching the spectrum of $y[n]$ to the spectrum of the input signal $x[n]$. We first try an adaptive delta modulator (DM).

For the Δ modulator shown in Fig. 1, if the step is small and the input signal has a steep slope, the Δ modulator may loose tracking and a slope overload occurs. A large step size will incur a large granular noise. To avoid this problem, adaptive $Delta$ modulator should be used [7]. Fig.4 shows a first-order digital adaptive Δ system, and the adaptive step size is given as [7]:

$$\Delta_n = \Delta_{n-1} K^{y(n)y(n-1)}. \tag{4}$$

where $K > 1$ is constant that can be arranged to minimize error. The quantization process of the LFM is shown in Fig. 5. The spectrum of $y[n]$ is getting closer to the spectrum of $x[n]$ as compared to non-adaptive Δ system over the baseband, but the distortion is large, thus an integrator is still necessary at the decoder.

The above analysis shows that the Δ modulator is sensitive to the rate of change of the signal.

3. SPECTRAL ANALYSIS OF THE $\Sigma\Delta$ MODULATOR

Δ modulation requires two integrators for the modulation and demodulation processes as shown in Fig. 1. Since integration is a linear operation, the second integrator can be moved before the modulator and becomes a preprocessing integrator. This will then be the configuration of $\Sigma\Delta$ modulator with two integrators. Again, based on the linear property of integration, the two integrators in Fig. 1 can be combined into a single integrator. A general configuration of a 1-bit $\Sigma\Delta$ modulator is shown in Fig. 6.

We again implement a digital version of Fig. 6 and show that a $\Sigma\Delta$ modulation system is a better solution for spectral matching. $\Sigma\Delta$ modulators encode the integral of the signal and thus their performance is insensitive to the rate of change of the signal.

Fig. 7 shows that the spectrum of the LFM $x[n]$ is

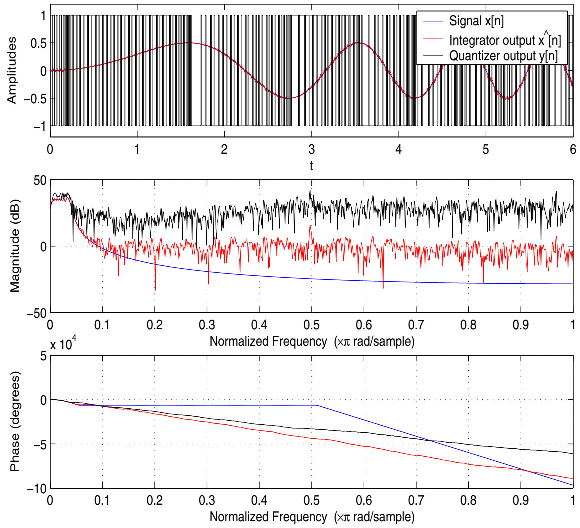

Fig. 5. Quantizing an LFM by adaptive Δ modulator.

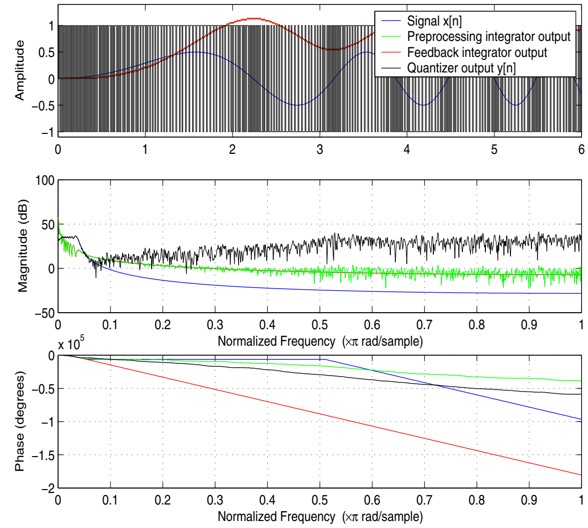

Fig. 7. Quantizing an LFM by the $\Sigma\Delta$ modulator with two integrators: $x[n]$, preprocessing integrator output, feedback integrator output, $y[n]$ and their corresponding spectra.

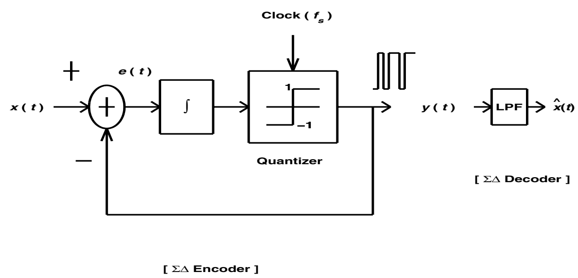

Fig. 6. First-order digital 1-bit $\Sigma\Delta$ modulator and demodulator.

squashed to lower frequencies by the preprocessing integrator and thus we have a situation similar to that in Fig. 1 and Fig. 2. The spectrum of the encoder output $y[n]$ matches the spectrum of the input of the encoder $x[n]$ over the bandwidth of interest, and hence only a LPF is needed in the decoder. The $\Sigma\Delta$ modulator is less sensitive to the quantization step and the sampling rate.

It should be also interesting to investigate the spectra of the $\Sigma\Delta$ modulator with one integrator as in Fig. 6. As shown in Fig. 8, there is no "spectrum squash" observed and the bandwidth of the output of the integrator is almost the same as that of the input signal $x[n]$. As in the case of two-integrator $\Sigma\Delta$, the spectrum of the input signal $x[n]$ matches that of the quantizer output $y[n]$ inside the bandwidth of interest, thus the decoder only needs a LPF to recover the input signal.

4. NOISE SHAPING OF THE $\Sigma\Delta$ MODULATOR

The noise shaping ability of an oversampling $\Sigma\Delta$ modulator allows the input signal of interest (baseband) to pass essentially unfiltered through the modulator but high-pass filters the quantization noise. For the ease of analysis of the important characteristics such as noise shaping, the nonlinearity of the quantizer in the $\Sigma\Delta$ modulator is approximated by an analytical linear model. There are arguments about the validity of the white noise linear model [4]. We investigate under what condition the linear model is valid. In the literature, Fig. 6 is approximated by a linear model to make the analysis tractable, i.e., the quantizer is linearized by using an input-independent additive white noise model, and the modulator output is given by [4] [6]:

$$Y(z) = X(z)z^{-1} + E(z)(1 - z^{-1}) \qquad (5)$$

where $X(z)$, $Y(z)$, and $E(z)$ are the z-transforms of the input, the output, and the quantization error, respectively. If we let $H_x = z^{-1}$ and $H_e(z) = (1 - z^{-1})$, the output is just a delayed version of the signal plus quantization noise that has been shaped by a first-order z-domain differentiator or a high-pass filter. This process is known as the "noise shaping". The corresponding time-domain version of the modulator output is

$$y[n] = x[n-1] + e[n] - e[n-1] \qquad (6)$$

where the $e[n] - e[n-1]$ term is the first-order difference of $e[n]$.

The transfer function $H_e(z) = (1 - z^{-1})$ is also called *noise transfer function* (NTF) $N(z) = (1 - z^{-1})$, and the magnitude of the NTF can be found out by letting $z = e^{j2\pi f/f_s}$ and we have

$$|N(f)| = 2\sin(\pi f/f_s). \qquad (7)$$

From Eq. 5 and Eq. 6, we obtain $\mathcal{Z}(e[n] - e[n-1]) = \mathcal{Z}(y[n] - x[n-1]) = E(z)(1 - z^{-1})$. This provides a way to evaluate the spectrum of $E(z)(1 - z^{-1})$ from the difference of the input $x[n-1]$ and the output $y[n]$. Again, we use the LFM as the input signal $x[n]$. Fig. 9 shows a comparison of the magnitude spectra for $N(f)$ and simulated power spectral density (PSD) of $E(z)(1 - z^{-1})$ for the $\Sigma\Delta$ modulator shown in Fig.

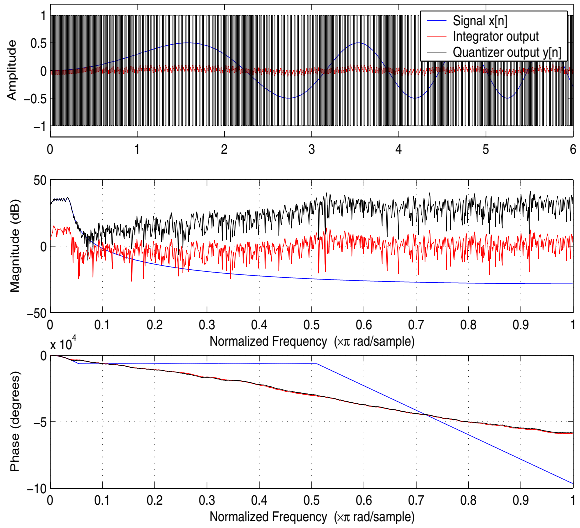

Fig. 8. Quantizing an LFM by the $\Sigma\Delta$ modulator with one integrators: $x[n]$, Integrator output, $y[n]$ and their corresponding spectra.

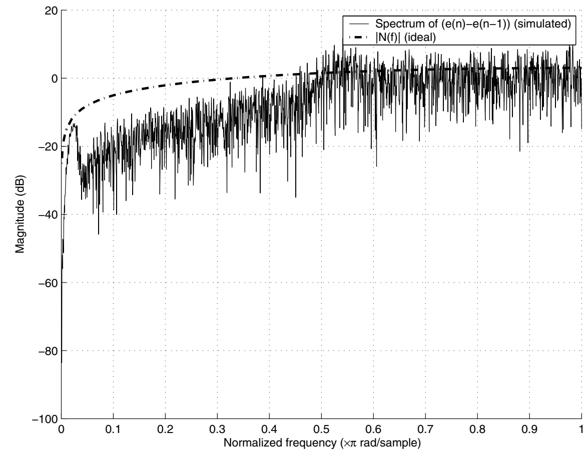

Fig. 9. Noise shaping of the first-order $\Sigma\Delta$ modulator. Compare between $N(f)$ and the power spectrum density of $(y[n]-x[n-1])$ or $(e[n]-e[n-1])$. The dotted line shows the first-order noise-shaping characteristic as predicted by Eq. 5.

6. Apparently, the match between the two curves in shape means $E(z)$ = constant, and hence $e[n]$ is a white noise process. In our simulation with an LFM input, increasing the duration of the LFM means a better match of the two curves (up to a scaling factor). The PSD of $(y[n] - x[n-1])$ in Fig. 9 is obtained with a sample size of 2^{12} and OSR $= 25$. The *noise shaping* is observed as the noise over the band of interest is significantly attenuated and is high-pass filtered outside the band of interest . In other words, the noise introduced by the quantization is pushed to higher frequencies which can be easily filtered out by a low pass filter (LPF) in DAC. Simulation also shows that with a higher OSR, the noise over the band of interest will be further attenuated. We conclude that at least in this case the linear model with the quantization error modelled as a white noise source is a valid assumption.

5. CONCLUSION

A linear FM (LFM) signal is applied as a test signal at the input of digital Σ and $\Sigma\Delta$ modulators to reveal the relationship of the spectra at different stages. By squashing the spectrum of the signal using the $\Sigma\Delta$ modulator, a simpler decoder and a better performance is achieved for the $\Sigma\Delta$ modulator. We see that for the same parameters such as the sampling frequency and quantization step, the $\Sigma\Delta$ modulator outperforms the adaptive Δ modulator.

The noise shaping ability and the validity of the linear model of the $\Sigma\Delta$ modulator are also investigated.

Similar principle may be applied to other configurations of the $\Sigma\Delta$ modulator in future works.

6. REFERENCES

[1] Ian Galton, “Delta-Sigma Data Conversion in Wireless Transceivers,” *IEEE Trans. Microwave Theory and Techniques*, vol. 50, no. 1, pp. 302-315, Jan. 2002.

[2] Zhongming Shi, “Sigma-Delta ADC and DAC for Digital Wireless Communication,” *IEEE Radio Frequency Integrated Circuits Symposium*, 1999.

[3] R. W. Stewart and E. Pfann, “Oversampling and Sigma-Delta Strategies for Data Conversion,” *IEE Electronics & Communication Engineering Journal*, pp. 37-47, Feb, 1998.

[4] Mucahit Kozak and Izzet Kale, *Oversampled Delta-Sigma Modulators, Analysis, Applications and Novel topologies*, Kluwer Academic Publishers, 2003.

[5] A. V. Oppenheim, R. W. Schafer and J. R. Buck, *Discrete-Time signal Processing*, Prentice Hall, 1999, 2nd ed.

[6] P. M. Aziz, H. V. Sorensen and J. V. Der Spiegel, “An Overview of Sigma-Delta Converters,” *IEEE Signal Processing Magzine*, Jan, 1996.

[7] J. Proakis, *Digital Communications*, 4-th Edition, Mc Graw-Hill, 2001.